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# Regular quantum graphs

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## Abstract

We introduce the concept of regular quantum graphs and construct connected quantum graphs with discrete symmetries. The method is based on a decomposition of the quantum propagator in terms of permutation matrices which control the way incoming and outgoing channels at vertex scattering processes are connected. Symmetry properties of the quantum graph as well as its spectral statistics depend on the particular choice of permutation matrices, also called connectivity matrices, and can now be controlled easily. The method may find applications in the study of quantum random walks and may also prove to be useful in analysing universality in spectral statistics.

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## 1. Introduction

The study of quantum graphs has become popular in a number of fields in quantum mechanics ranging from molecular physics and the physics of disordered systems to quantum chaos and quantum computation (see e.g. [Ku02]). Quantum graphs serve as computationally inexpensive models with the ability to mimic a variety of features also present in more realistic quantum systems. For example, the 20 year old conjecture by Bohigas, Giannoni and Schmidt (BGS) [BGS84], stating that the spectral statistics of quantum systems whose classical limit is chaotic follow those of random Hermitian or unitary matrices in the semiclassical limit, is well reproduced on quantum graphs [KS97, KS99].

In this paper, we will address two fundamental, but seemingly disconnected questions related to quantum graphs, namely, we will look at (1) ways to introduce symmetries on connected quantum graphs and investigate (2) the degree of complexity or randomness necessary on a quantum graph to fall within the universal regime of random matrix statistics.

The first point has hardly been addressed in the context of quantum graphs. Symmetries on quantum graphs play an important role in studies on quantum random walks considered recently in the context of quantum computation (see e.g. [Ke03]). Speed up of mixing parameters of quantum random walks over classical random walks found on certain graphs is

indeed related to interference effects due to symmetries in the quantum propagation. We will suggest a method for imposing a large class of symmetries on certain types of graphs which has potential applications in the design of effective quantum random walks.

It is furthermore expected that symmetries on graphs will have a profound influence on the statistical properties of spectra of quantum graphs. The existence of discrete symmetries and associated ‘good quantum numbers’ on connected quantum graphs is expected to lead to deviations from random matrix results.

The second point addresses the range of validity of the BGS-conjecture. It is widely believed that the spectra of unitary propagators on quantum graphs follow random matrix statistics if the correlation exponents of an underlying stochastic dynamics are bound away from zero in the limit of large graph sizes and the arcs lengths of the graph is incommensurate [Ta01]. We will argue here that the latter condition can be considerably relaxed and that, in the context of regular graphs, the existence or absence of random matrix statistics is related to the commutativity properties of certain sets of *connectivity matrices* to be defined in detail later. Similar results for the spectra of Laplacians of regular graphs have been reported in [JMRR99]. A related discussion of spectra of adjacency matrices of Cayley graphs of certain groups can be found in [Te03].

We start by briefly reviewing the notion of a quantum graph. A quantum graph is given by an underlying graph  $G$  and a set of local scattering matrices at the vertices as well as a set of arc lengths. A (finite) *directed graph* or *digraph* consists of a finite set of *vertices* and a set of ordered pairs of vertices called *arcs*. We denote by  $V^G$  and  $E^G$  the set of vertices and the set of arcs of the digraph  $G$ , respectively. Given an ordering of the vertices, the *adjacency matrix* of a digraph  $G$  on  $n$  vertices, denoted by  $A^G$ , is the  $(n \times n)(0, 1)$ -matrix where the  $ij$ th element is defined by

$$A_{ij}^G := \begin{cases} 1 & \text{if } (i, j) \in E^G \\ 0 & \text{otherwise.} \end{cases}$$

An *undirected graph* (for short, *graph*) is a digraph whose adjacency matrix is symmetric. The *line digraph* of a digraph  $G$ , denoted by  $LG$ , is defined as follows (see e.g. [BG01]):  $V^{LG} = E^G$  and, given  $(h, i), (j, k) \in E^G$ , the ordered pair  $((h, i), (j, k)) \in E^{LG}$  if and only if  $i = j$ .

A *quantum graph* associated with a digraph  $G$  on  $n$  vertices may then be defined in terms of a set of unitary vertex scattering matrices  $\sigma^{(j)}$  on vertices  $j = 1, \dots, n$  and a set of arc-lengths  $L_{(i,j)}$  defined for every arc  $(i, j) \in E^G$ . Waves propagate freely along the directed arcs; transitions between incoming and outgoing waves at a given vertex  $j$  are described by the scattering matrix  $\sigma^{(j)}$ . The two sets specify a unitary propagator of dimension  $n_E = |E^G|$  defining transitions between arcs  $(i, j), (i', j') \in E^G$  which has the form [KS97]

$$S^G = DV \quad \text{with} \quad D_{(i,j)(i',j')} = \delta_{i,i'} \delta_{j,j'} e^{ikL_{(i,j)}}$$

where  $k$  is the wave number and

$$V_{(i,j)(i',j')} = A_{(i,j)(i',j')}^{LG} \sigma_{ij'}^{(j)} \quad \text{with} \quad A_{(i,j)(i',j')}^{LG} = \delta_{j,i'}.$$

The local scattering matrices  $\sigma^{(i)}$  depend on the boundary conditions and local potentials at the vertex  $i$  which we do not want to specify here any further. For our purpose, we may regard the  $\sigma^{(i)}$  as arbitrary unitaries. Let  $d_i^-$  and  $d_i^+$  be the number of incoming and outgoing arcs of a vertex  $i$ , respectively. A sufficient and necessary condition for a digraph  $G$  to be quantizable in the way given above is then, that for every vertex  $i \in V^G$ ,  $d_i^+ = d_i^- = d_i = \dim \sigma^{(i)}$  [PTZ03, S03]. This means in particular that if  $G$  is an undirected graph then it is quantizable.

The ‘classical’ dynamics corresponding to a quantum graph defined by a unitary propagator  $S^G$  is given by a stochastic process with transition matrix  $T$

$$T_{ij} = |S_{ij}^G|^2 = |V_{ij}|^2.$$

Note that both the quantum mechanics as well as the associated stochastic dynamics relate to transitions between arcs and are thus defined on the line digraph of  $G$ .

The paper is organized as follows. In section 2, we will introduce the notion of regular quantum graphs and discuss a factorization of the propagator in terms of connectivity matrices for a special class of such graphs. In section 3, we relate the existence or absence of symmetries on a connected regular quantum graph to properties of the connectivity matrices. We discuss some specific examples for completely connected graphs including statistical properties of the spectra in section 4. In section 5, we show numerically that by inscribing a *single*  $(2 \times 2)$  unitary matrix into a large regular quantum graph one still obtains random matrix statistics despite huge degeneracy in the set of arc lengths and scattering matrices for a generic choice of connectivity matrices.

## 2. Regular quantum graphs

We will implement symmetries on quantum graphs for which the wave dynamics at a given vertex of the digraph is ‘locally indistinguishable’ from that of any other vertex of the graph. We will restrict ourselves to wave dynamics on  $d$ -regular digraphs. Recall that a digraph  $G$  is said to be  $d$ -regular if, for every vertex  $i \in G$ ,  $d_i^+ = d_i^- = d$  and thus  $|E^G| = nd$ . Extending the concept of local indistinguishableness to quantum graphs, we will consider quantum graphs on  $d$ -regular digraphs with local  $(d \times d)$  scattering matrices  $\sigma^{(i)}$  and set of outgoing arc lengths  $L_{(i,j)}$  being identical at every vertex  $i$  up to permutations of the incoming or outgoing channels. That is, there are  $(d \times d)$  unitary matrices  $\sigma$  and  $D(k)$  with  $D_{ij}(k) = \delta_{i,j} \exp(ikL_i)$  and local permutation matrices  $q^{(i)}, p^{(i)}$ , such that

$$\sigma^{(i)} = p^{(i)} \sigma q^{(i)} \quad \text{and} \quad D^{(i)}(k) = p^{(i)} D(k) (p^{(i)})^{-1}.$$

Combining the local matrices  $\sigma$  and  $D(k)$  to a single matrix  $C(k) = D(k)\sigma$ , we obtain

$$C^{(i)}(k) = D^{(i)}(k)\sigma^{(i)} = p^{(i)} C q^{(i)}. \tag{1}$$

We call a quantum graph with these properties a *regular quantum graph*. The matrix  $C$  is called the *coin* in the context of quantum random walks on graphs [Ke03].

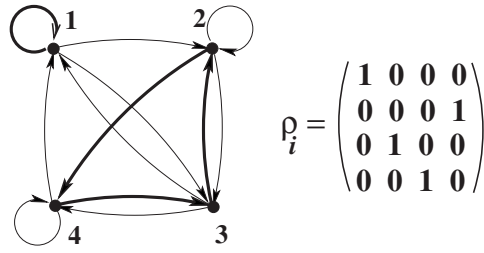
We denote by  $J_n$  the  $(n \times n)$  matrix with all elements being equal to 1 and  $I_n$  is the identity matrix. The following observation will be useful in what follows (see also [S03a]):

**Proposition 1.** *Let  $A^G$  be the adjacency matrix of a  $d$ -regular digraph  $G$ . The adjacency matrix of  $LG$  has up to reordering the arcs the form*

$$A^{LG} = \left( \bigoplus_{i=1}^d \rho_i \right) \cdot (J_d \otimes I_n) \tag{2}$$

where the matrices  $\rho_i$  of dimension  $n$  have entries 0 or 1 only and  $\rho_i e_j = e_k \forall i, j$  and some  $k \in \{1, \dots, n\}$ ; here  $e_j$  is the  $j$ th vector in the standard basis. In addition, we have

$$\sum_{i=1}^d \rho_i = A^G. \tag{3}$$



**Figure 1.** A 3-regular graph of size 4 together with a possible connectivity matrix relating an outgoing arc to every vertex.

**Remark.** A given matrix  $\rho_i$  assigns to every vertex  $j$  a specific arc  $(j, \rho_i(j))$  with  $\rho_i(j) = \rho_i e_j$ , see figure 1. Note that, for  $d > 1$ , the choice of matrices  $\rho_i$  is in general not unique and that the  $\rho_i$  do not need to be invertible. Different sets of  $\rho_i$  fulfilling the conditions in the proposition give rise to adjacency matrices of the line digraph which are equivalent up to permutations in the ordering of the arcs.

**Proof.** Condition equation (3) ensures that  $(j, \rho_i(j)) \in E^G$  for every  $i$  and  $j$ ; writing equation (2), we obtain

$$\bigoplus_{i=1}^d \rho_i = \begin{bmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_d \end{bmatrix} \quad \text{and thus} \quad \left( \bigoplus_{i=1}^d \rho_i \right) \cdot (J_d \otimes I_n) = \begin{bmatrix} \rho_1 & \rho_1 & \cdots & \rho_1 \\ \rho_2 & \rho_2 & \cdots & \rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_d & \rho_d & \cdots & \rho_d \end{bmatrix}.$$

The choice of matrices  $\rho_i$  fixes the ordering of the arcs according to  $(1, \rho_1(1)), (2, \rho_1(2)), \dots, (n, \rho_1(n)), (1, \rho_2(1)), \dots, (n, \rho_2(n)), \dots, (1, \rho_d(1)), \dots, (n, \rho_d(n)).$  (4)

The non-zero matrix elements of  $A^{LG}$  as defined in (2) thus refer to the allowed transitions  $(i, \rho_j(i)) \rightarrow (\rho_j(i), \rho_k(\rho_j(i)))$  with  $j, k = 1, \dots, d$  and  $i = 1, \dots, n$  in the line digraph of  $G$ . □

We pointed out in the introduction that the wave propagation on a quantum graph actually lives on the line digraph of the underlying digraph. Generalizing (2) to describe unitary propagation on digraphs, we write

$$S^G = \left( \bigoplus_{i=1}^d \rho_i \right) \cdot (C \otimes I_n) = \begin{bmatrix} C_{11}\rho_1 & C_{12}\rho_1 & \cdots & C_{1d}\rho_1 \\ C_{21}\rho_2 & C_{22}\rho_2 & \cdots & C_{2d}\rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ C_{d1}\rho_d & C_{d2}\rho_d & \cdots & C_{dd}\rho_d \end{bmatrix} \quad (5)$$

with  $C$  being the unitary  $(d \times d)$  coin and the matrices  $\rho_i$  fulfil the condition (3). We have to add the additional constraints here that the  $\rho_i$  are invertible, that is, that they are permutation matrices. The condition is necessary to ensure that  $S^G$  is unitary. We will refer to the permutations  $\rho_i$  as the *connectivity matrices* in what follows, see (1).  $S^G$  satisfies all the properties of a regular quantum graph as defined above. The matrix  $C$  is in particular the coin from which the local scattering matrices  $C^{(j)}$  at vertices  $j$  can be deduced. One obtains

$$C_{kl} = C_{\rho_k^{-1}(j)\rho_l(j)}^{(j)}$$

The connectivity matrices  $\rho_k$  and  $\rho_l$  thus specify the pair of arcs related through the transition  $C_{kl}$  at a given vertex  $j$ .

**Remark.** In contrast to (2) where different decomposition of  $A^G$  of the form (3) lead to equivalent adjacency matrices (up to reordering the arcs), this is no longer the case for (5). Different sets of connectivity matrices lead here to different regular quantum graphs which may have very different spectral properties as will be discussed in the next section.

**Remark.** Note that not all regular quantum graphs can be written in the form (5). Any pair of permutation matrices  $P$  and  $Q$  leaving the adjacency matrix of a line graph  $A^{LG}$  of a  $d$ -regular graph  $G$  invariant, that is  $QA^{LG}P = A^{LG}$ , transform an associate propagator of a regular quantum graph,  $S^G$ , into a propagator of a  $d$ -regular, albeit different, quantum graph  $\tilde{S}^G = PS^GQ$ . If  $S^G$  is of the form (5), one easily finds permutations  $P$  and  $Q$  such that  $\tilde{S}^G$  is not of this form.

So far we have considered general regular digraphs. In the special case where the underlying graph is undirected, it is natural to consider associated time-reversal symmetric regular quantum graphs; that is, regular quantum graphs for which for every (wave) path there exists an equivalent time-reversed path undergoing the same transitions. A time-reversal symmetric unitary propagator of the form (5) for an undirected regular graph can be constructed by choosing symmetric coin and connectivity matrices, that is,

$$C = C^T \quad \text{and} \quad \rho_i = \rho_i^T \quad \text{for every } i = 1, \dots, d.$$

Note that the symmetry conditions for the connectivity matrices severely limit the choice of possible graphs and decompositions.

### 3. Symmetries on regular quantum graphs and spectral decompositions

We first note that if a regular quantum graph can be written in the form (5) and there exists an invertible  $(n \times n)$  matrix  $\pi$  such that

$$[\pi, \rho_i] = 0 \quad \text{for every } i = 1, \dots, d \tag{6}$$

then

$$[P, S^G] = 0 \quad \text{with} \quad P = (I_d \otimes \pi) \tag{7}$$

independent of the choice of the coin  $C$ . The result follows immediately from

$$[(C \otimes I_n), P] = 0.$$

It is obvious that the condition (6) implies  $[\pi, A^G] = 0$ .

The above property enables us to study certain symmetries of quantum graphs in terms of the symmetries of the connectivity matrices only. Given a  $d$ -regular digraph  $G$ , we can in general find many sets of connectivity matrices which sum up to  $A^G$  and which may have very different symmetry properties. Or if one is interested in quantum graphs with specific symmetries one may start from a set of connectivity matrices  $\rho_i$  in order to construct quantum graphs with desired properties. We consider various scenarios here and give some specific examples in the next section.

#### 3.1. The Abelian case: $[\rho_i, \rho_j] = 0$

In the special case when all connectivity matrices commute, every  $\rho_i$  acts as a symmetry  $\pi$ . The spectrum of  $S^G$  can then be decomposed into  $n$  sub-spectra of dimension  $d$ .

**Proposition 2.** Let  $S^G$  be of the form

$$S^G = \left( \bigoplus_{i=1}^d \rho_i \right) \cdot (C \otimes I_n)$$

where the  $(n \times n)$  connectivity matrices  $\rho_i$  fulfil  $[\rho_i, \rho_j] = 0$ ,  $A_G = \sum_{i=1}^d \rho_i$  is the adjacency matrix of a  $d$ -regular digraph and  $C$  is a  $(d \times d)$  unitary matrix. Let  $u$  be the  $(n \times n)$  unitary matrix simultaneously diagonalizing the  $\rho_i$ , that is,

$$u^\dagger \rho_i u = \bigoplus_{m=1}^n e^{i\varphi_m^i} \quad i = 1, \dots, d$$

and  $\varphi_m^i$  is the  $m$ th eigenphase of the connectivity matrix  $\rho_i$  where the order is determined by the transformation  $u$ . The spectrum of  $S^G$ ,  $sp(S^G)$ , is then

$$sp(S^G) = sp(S_1^G) \uplus sp(S_2^G) \uplus \dots \uplus sp(S_n^G)$$

where

$$S_m^G = \left( \bigoplus_{i=1}^d e^{i\varphi_m^i} \right) \cdot C. \quad (8)$$

**Proof.** Define  $U = (I_d \otimes u)$  and note that

$$[U, (C \otimes I_n)] = 0.$$

Thus

$$U^\dagger S^G U = \left( \bigoplus_{i=1}^d \bigoplus_{m=1}^n e^{i\varphi_m^i} \right) \cdot (C \otimes I_n).$$

There exist permutation matrices  $P$  such that  $P^T (C \otimes I_n) P = (I_n \otimes C)$  and thus

$$P^T U^\dagger S^G U P = \left( \bigoplus_{i=1}^n S_m^G \right)$$

is indeed block diagonal of the form stated in the proposition.  $\square$

**Remark.** Note that the decomposition is independent of the coin  $C$ .

It can be shown [ST04] that a set of commuting connectivity matrices of a connected graph  $G$  always form a subset of the regular (permutation) representation of an Abelian group and the underlying symmetry of the corresponding quantum graph is given by that group. (The commutativity of the  $\rho_i$  does in fact imply that  $G$  is a Cayley digraph of an Abelian group; it must therefore have the form of a discretized torus). The sub-spectra obtained from  $S_m^G$  may then be characterized in terms of the eigenbasis of the generators of the Abelian group represented by the connectivity matrices. Let  $a_1, \dots, a_r$  be the generators of such an Abelian group,

$$\text{with } a_i^{n_i} = id \quad \text{and} \quad \prod_{i=1}^r n_i = n \quad \text{where } n_i \geq 2.$$

The eigenbasis of the connectivity matrices may then be written in Dirac notation as  $|m_1 \dots m_r\rangle$  with  $m_i = 1, \dots, n_i$  and the sub-spectra obtained from (8) are characterized by a set of  $r$  ‘quantum numbers’  $S_{m_1, \dots, m_r}^G$ . Such a regular quantum graph is thus a discretized version of quantum systems whose underlying classical dynamics has  $r$ -integrals of motion in involution.

Some additional degrees of freedom are represented by the coin  $C$  which may or may not be related to classical chaotic dynamics depending on the properties of  $C$  and the group.

3.2. *Partial symmetries:*  $[\pi, \rho_i] = 0$ , but  $[\rho_i, \rho_j] \neq 0$

Next we consider the case that a symmetry  $\pi$  exists with  $[\pi, \rho_i] = 0$  for all  $i \in V^G$ , but  $[\rho_i, \rho_j] \neq 0$  for some  $i, j = 1, \dots, d$ . This implies that  $\pi$  has degenerate eigenvalues.  $\pi$  could for example represent a  $C_2$  symmetry of the graph, that is,  $\pi^2 = I_n$  with eigenvalues  $\pm 1$  only.

Let us assume that  $\pi$  has  $r < n$  distinct eigenvalues  $\lambda_i, i = 1, \dots, r$ , each with multiplicity  $n_i$  with

$$\sum_{i=1}^r n_i = n.$$

Let  $u$  be a unitary matrix diagonalizing  $\pi$  in the form

$$u^\dagger \pi u = \bigoplus_{i=1}^r \lambda_i I_{n_i}$$

$u$  then brings  $\rho_i$  into block-diagonal form, that is,

$$u^\dagger \rho_i u = \bigoplus_{j=1}^r \tilde{\rho}_i^{(j)}$$

with  $\dim \tilde{\rho}_i^{(j)} = n_j$ . The spectrum of  $S^G$  is now decomposed in the following way:

**Proposition 3.** *Let  $S^G$  be of the form (5) and the matrices  $\rho_i, \pi$  have the properties as described above;  $sp(S^G)$  is then of the form*

$$sp(S^G) = sp(S_1^G) \uplus sp(S_2^G) \uplus \dots \uplus sp(S_r^G)$$

with

$$S_m^G = \left( \bigoplus_{i=1}^d \tilde{\rho}_i^{(m)} \right) \cdot (C \otimes I_{n_m}) \quad \text{where } m = 1, \dots, r. \tag{9}$$

The proof is similar to the one of proposition 2.

The decomposition is again independent of the coin  $C$ , but the sub-spectra are now of dimension

$$\dim S_m^G = n_m d.$$

There is a trivial symmetry independent of the particular choice of the  $\rho_i$  related to the fact that every permutation matrix has an eigenvalue 1 with corresponding eigenvector  $(1, 1, \dots, 1)^T$ . The symmetry  $\pi$  in question has the form

$$\pi_{ij} = \frac{2}{n} - \delta_{i,j}$$

having two distinct eigenvalues  $\pm 1$  and the eigenvalue  $-1$  has geometric multiplicity  $n - 1$ . As a consequence any  $S^G$  can be block diagonalized containing  $C$  as a  $(d \times d)$  block, and thus

$$sp(C) \subset sp(S^G).$$



#### 4. Some examples for $A^G = J_n$

When constructing particular examples, it is useful to start with the completely symmetric graph, namely that of a *fully connected graph*. This graph, also called the *complete graph*, has adjacency matrix  $J_n$ . As  $[P, J_n] = 0$  for every permutation matrix  $P$  of size  $n$ , we may indeed construct regular quantum graphs of degree  $d = n$  with whatever finite symmetry we want. In addition, we can make use of the fact that if  $\Gamma$  is a finite group of order  $n$  and the  $(n \times n)$  permutation matrices  $\rho_i$  form a regular representation of  $\Gamma$  then

$$\sum_{i=1}^n \rho_i = J_n. \quad (10)$$

We can thus implement the group properties of any finite group on a regular quantum graph by choosing the regular representations of that group as the connectivity matrices. In what follows we will consider various decompositions of  $J_n$  and see how they affect statistical properties of the quantum graph spectra.

##### 4.1. The cyclic group $\mathbb{Z}_n$

The simplest Abelian group is the cyclic group  $\mathbb{Z}_n$ . The regular representations  $\rho_i$  are of the form

$$(\rho_j)_{kl} = \delta_{k, (l+j) \bmod n} \quad \text{with eigenvalues} \quad \chi_m^j = e^{2\pi i \frac{jm}{n}} \quad \text{where } j, m = 1, \dots, n.$$

Here,  $\rho_j = (\rho_1)^j$  and  $\rho_n = I_n$ . In order to construct regular quantum graphs with circular symmetry independent of the coin  $C$  we use as connectivity matrices the regular representation of  $\mathbb{Z}_n$ . The spectrum of the quantum graph can then be decomposed into the sub-spectra given by

$$S_m^G = \left( \bigoplus_{j=1}^n e^{2\pi i \frac{jm}{n}} \right) \cdot C.$$

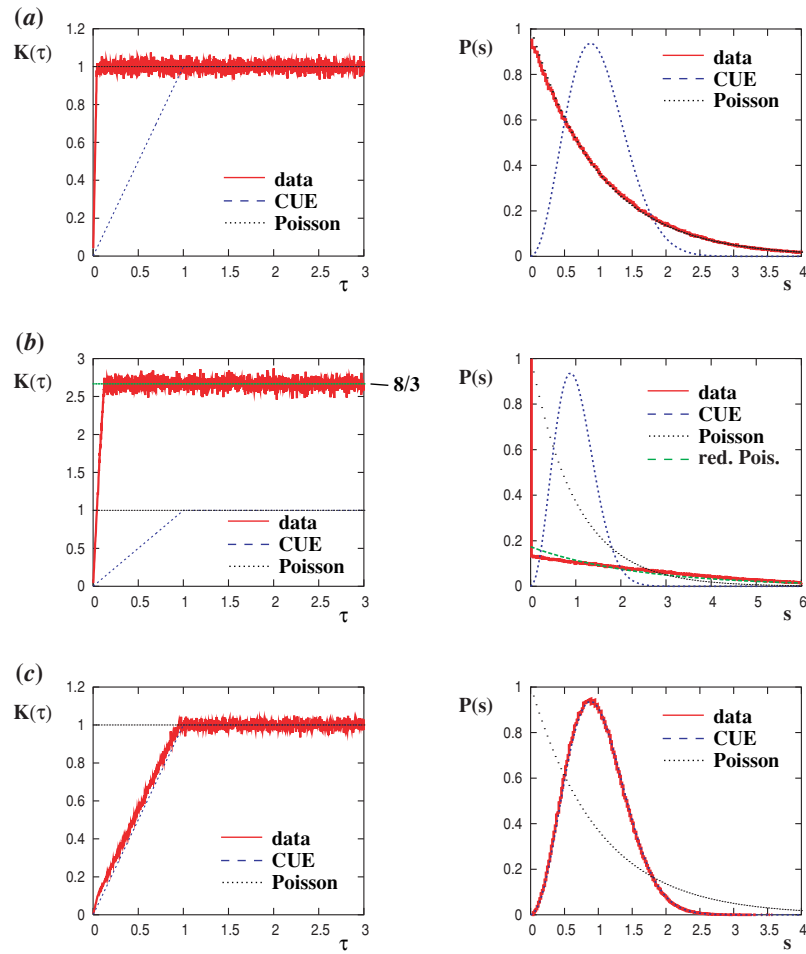
The eigenvalues are characterized in terms of two quantum numbers, an angular momentum quantum number  $m$  and a second quantum number  $r$ , say, counting the eigenvalues in each  $m$  manifold. If the spectra for different  $m$  are uncorrelated, one expects Poisson statistics of the total spectrum in the limit  $n \rightarrow \infty$ .

Figure 2(a) shows spectral properties of  $S^G$  with  $n = 24$ , that is,  $\dim S^G = 576$ . We plot here the nearest-neighbour spacing (NNS) distribution  $P(s)$  and the form factor  $K(\tau)$ , the Fourier transform of the two-point correlation function. The coin is of the form (1) where the local scattering matrix  $\sigma$  is taken randomly from a *CUE*-ensemble and the arc lengths entering the diagonal matrix  $D$  are chosen independently and identically distributed in  $[0, 1]$ . The average is, for a fixed choice of coin and arc lengths, taken by averaging over the wavelength  $k$  [Ta01].<sup>3</sup> The numerical results shown in figure 2(a) indeed suggest Poisson-statistics apart from deviations in the form factor on scales  $\tau \leq 1/n$  due to the ‘chaotic nature’ of the coin.

<sup>3</sup> Alternatively, one can consider the spectrum being the ‘resonances’ of the quantum graph at wave numbers given by

$$\det(1 - S^G(k)) = 0.$$

Both approaches are equivalent under very general conditions [BK99].



**Figure 2.** Form factor  $K(\tau)$  and nearest-neighbour spacing distribution  $P(s)$  for (a)  $\rho_i$  are the regular representation of the cyclic group  $\mathbb{Z}_{24}$ ; (b)  $\rho_i$  represent the symmetric group  $S_4$ ; (c) a generic set  $\rho_i$  without non-trivial symmetries. The dashed curve in (b) labelled ‘red. Poiss.’ corresponds to a distribution of degenerate levels being Poisson distributed otherwise.

4.2. The non-Abelian case: the symmetric group  $S_4$

Next, we consider a specific example of a non-Abelian group, namely the symmetric group  $S_4$  with  $n = 24$  elements; we will discuss spectral properties of general groups elsewhere [ST04]. The regular representation of  $S_4$  can be decomposed in terms of its irreducible representations (for short *irreps*); each  $\rho_i$  contains each  $d$ -dimensional irrep exactly  $d$  times. The group  $S_4$  has two one-dimensional, one two-dimensional and two three-dimensional irreps, such that

$$2 \times 1^1 + 1 \times 2^2 + 2 \times 3^3 = 24.$$

Denote the irreps of the group element  $i \in S_4$  as

$$\tilde{\rho}_i^{(1,1)}, \tilde{\rho}_i^{(1,2)}, \tilde{\rho}_i^{2,1}, \tilde{\rho}_i^{(3,1)}, \tilde{\rho}_i^{(3,2)}$$

with  $\dim \tilde{\rho}_i^{(d,l)} = d$  and the index  $l$  counting different irreps of the same dimension; there

exists then a transformation  $u$  such that

$$u^\dagger \rho_i u = \left( \bigoplus_{l=1}^2 \tilde{\rho}_i^{(1,l)} \right) \oplus (I_2 \otimes \tilde{\rho}_i^{(2,1)}) \oplus \left( \bigoplus_{l=1}^2 (I_3 \otimes \tilde{\rho}_i^{(3,l)}) \right).$$

The connectivity matrices  $\rho_i$  are thus of the form as discussed in section 3.2. Note that the sub-matrices  $S_{d,l}^G$  related to  $d$ -dimensional irreps now occur  $d$  times in the decomposition. We thus have five independent sub-spectra, two of dimension 24, one of dimension 48 and two of dimension 72 of which the latter are of multiplicity two and three, respectively. The huge degeneracy in the spectra can clearly be seen in the spectral statistics; it is manifest in the peak at  $s = 0$  in  $P(s)$  (see figure 2(b) and leads to

$$K(\tau) = (2 \times 3^3 + 1 \times 2^3 + 2 \times 1^3) = 8/3 \quad \text{for } \tau > 3/24.$$

The spectra appear to be uncorrelated otherwise; note however, that the spectrum for each sub-matrix  $S_{d,l}^G$  alone is correlated following *CUE* statistics, which manifests itself in the deviations from purely Poisson behaviour in  $P(s)$  (cf dashed curve) as well as in the behaviour of the form factor for  $\tau \leq 3/24$  which is dominated by the sub-spectra of the three-dimensional irreps.

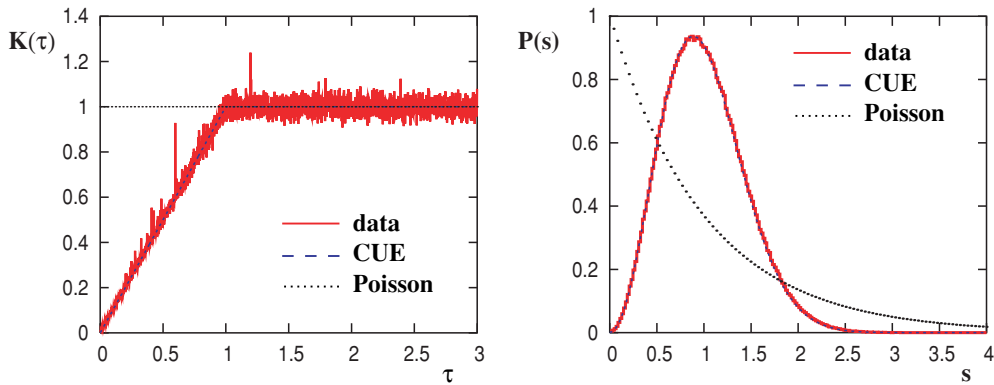
#### 4.3. The generic case: no symmetries

The overwhelming number of decompositions of the form (10) will, of course, have no common symmetry apart from the trivial symmetry discussed in section 3.2. Even though no further analytical results can be given in this case, a numerical study may reveal interesting insights into the range of validity of the RMT-regime. Figure 2(c) shows the level statistics of a regular quantum graph obtained from a fully connected graph for a generic choice of connectivity matrices. One indeed finds good agreement with random matrix theory for the *CUE*-ensemble. Deviations in the form factor for small  $\tau$  can be attributed to the fact that the spectrum of  $C$  is contained in the full spectrum. After removing this separable part of the spectrum as done for the NNS in figure 2(c) there is good agreement with random matrix results. It is worth keeping in mind that this is a highly non-random matrix; we are dealing here with the spectrum of the  $(n^2 \times n^2)$  unitary matrix  $S^G$  which has only  $n^3$  non-zero elements of which only  $n^2$  are independent. In particular, the arc lengths in the graph are not incommensurate, the  $n^2$  arcs in the graph share only  $n$  different lengths among them. Still, universality is obtained. The origin of the complexity in this type of quantum graphs is here clearly not due to the ‘randomness’ in the choice of the matrix elements but due to *the lack of a common symmetry* in the set of connectivity matrices.

### 5. Regular de Bruijn quantum graphs

The results in the last section suggest that spectral statistics of regular quantum graphs can to a large extent be controlled by properties of the permutation matrices  $\rho_i$  independent of the coin  $C$ . It is thus natural to ask whether we can reduce the dimension of the coin to its smallest possible value, namely  $\dim C = 2$ , by considering large 2-regular quantum graphs and still obtain random matrix correlations.

We can only expect random matrix statistics on a regular quantum graph if the corresponding quantum graph with randomly chosen arc-lengths falls into the random matrix category. We therefore need to consider 2-regular graphs leading to fast (classical) mixing and not for example diffusive networks such as ring graphs [SS00] exhibiting 1D-Anderson localization. The  $d$ -regular digraphs with the fastest mixing rates are the so-called  $d$ -ary



**Figure 3.** Form factor  $K(\tau)$  and nearest-neighbour spacing distribution  $P(s)$  for binary de Bruijn graph of order  $k = 9$  with  $\dim S^{B(2,9)} = 1024$ .

de Bruijn graphs of order  $k$ ,  $B(d, k)$ , being the  $(k - 1)$ th line graph generation of a complete graph of size  $d$ . That is

$$B(d, k) = L^{k-1}G \quad \text{with} \quad A^G = J_d$$

and  $L^k G$  is iteratively defined as  $L^k G = L(L^{k-1} G)$  [B46, FYA84]. De Bruijn graphs have size  $d^k$  and are equipped with a complete symbolic dynamics of order  $d$ . They play an important role in coding theory and parallel algorithms [SR91, SP89]. Numerical evidence suggests that quantum graphs based on de Bruijn digraphs with incommensurate arc lengths follow random matrix statistics in the limit of large graph sizes even for  $d = 2$  [Ta00, Ta01].

In figure 3, we show results for a regular quantum graph based on a binary de Bruijn graph  $B(2, k)$  with  $k = 9$  and quantum propagator

$$S^{B(2,k)} = (\rho_1 \oplus \rho_2) \cdot (C \otimes I_{2^k})$$

where  $\rho_1, \rho_2$  are permutation matrices with  $\rho_1 + \rho_2 = A^{B(2,k)}$  and  $C$  is a  $(2 \times 2)$  unitary matrix. The statistics in figure 3 are obtained by averaging over the space of  $(2 \times 2)$  unitaries with respect to the Haar measure. The connectivity matrices of dimension  $2^k$  have been chosen randomly. To avoid accidental symmetries, different sets of connectivity matrices have been produced and the statistics of the corresponding ensemble averages combined. The spectral statistics of these regular quantum graphs agrees again very well with CUE statistics. Recall that the unitary matrix  $S^{B(2,9)}$  of size  $2^{10} = 1024$  has  $2^{11}$  non-zero matrix elements which do, however, take only four different (complex) values! We have thus constructed extremely non-random matrices which still show universal random matrix statistics and have thereby shown that the BGS-conjecture is valid far beyond regimes previously thought to be included in the conjecture. Similar numerical results were also found for the spectra of the Laplacian of regular graphs [JMRR99]. A Laplacian on a  $d$ -regular undirected, loop-less graph  $G$  is  $\Delta = dI_n - A^G$  and is thus a symmetric matrix whose non-zero matrix elements take the values 1 or  $d$  only. When averaging over sets of  $d$ -regular graphs agreement with GOE statistics was found. This underlines once again that the origin of universality in spectral statistics lies not in the randomness of the matrix elements.

### 6. Conclusions

We introduce a decomposition of certain regular quantum graphs which separates the quantum propagator on a graph into a topological part containing the connectivity matrices and a part

containing the quantum scattering information at the vertices. This allows one to implement global symmetries on the graph by choosing the connectivity matrices according to desired symmetry properties. We demonstrate that the complexity in the quantum spectrum (which may be seen to take on its maximal value when the statistics coincides with RMT) can here be linked to the amount of complexity contained in the set of permutation matrices building up the quantum graph. We present examples, where for a given graph and a fixed coin matrix, we were able to construct anything from Poisson to RMT-statistics just by changing the set of connectivity matrices. By doing so, we leave the local properties of the graph invariant, but change the way in which incoming and outgoing channels between vertices are connected and thus the global structure of the wave dynamics. We take this concept to its extreme by demonstrating numerically that unitary matrices representing 2-regular quantum graphs whose non-zero matrix elements take only four different values still follow CUE statistics for de Bruijn graphs.

We believe that our results open up new perspectives in understanding universality in spectral statistics. It transforms the question from a continuous to an essentially discrete problem focusing on the way local scattering processes are connected and condensing the parameter space to an absolute minimum (namely four dimensional).

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